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THE IMPRESSION OF A STRIP-LIKE PUNCH INTO A HALF-SPACE WHICH DEFORMS EXPONENTIALLY†

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The contact problem of the impression of a strip-like punch into a half-space when there is a power-law relationship between the stress intensities and the strain rates [1, 2] is considered in formulating a non-linear theory of steady-state creep. Using the generalized principle of the superposition of displacements [1, 2], the problem is formulated in the form of the same integral equation as in an analogous problem in [3]. Unlike [3], a closed solution of this equation is constructed. This is achieved by using a spectral relationship for the integral operator which is generated in a finite interval by a symmetric difference kernel in the form of a MacDonald function.

A spectral relationship, containing spheroidal wave functions which are a generalization of Mathieu functions, was established in [4] using a method of the potential theory-type in another form. However, the eigenvalues were not obtained in explicit form and the complete conditions of the normalization of the eigenfunctions were not indicated. Meanwhile, analytic difficulties were encountered when calculating the eigenvalues in explicit form. The problem of deriving the above-mentioned spectral relationship in a constructive form using the method developed in [5] is therefore discussed again below. This relationship is completely established using generalized potential theory methods [3, 6, 7] associated with Euler–Poisson–Darboux equation and which, in the final analysis, differ from those in [4]. A relationship related to it is also established. This holds in additional semi-infinite intervals. Both of these can be used to solve many mixed problems in the theory of elasticity and mathematical physics.

An extensive class of similar spectral relationships for orthogonal polynomials and their numerous applications to mixed problems in the theory of elasticity were presented in [8, 9].

1. Let a punch which has the plan shape of a strip $\omega=\{z=0,\ -\infty<x<\infty,\ -a<y<a\}$, be displaced only translationally in a vertical direction and impressed into the half-space $z<0$, referred to a right rectangular system of coordinates $Oxyz$ under the action of certain moments and vertical forces $p_0(x,\ y)$, which are distributed along its length and possess a finite equivalent force P. We shall assume that the material of the half-space is incompressible and obeys a power law $\sigma_i=K\varepsilon_i^k$ $(0\leqslant h\leqslant 1)$, where σ_i and ε_i are the intensities of the stresses and strain rates, respectively, and K and h are physical constants [1, 2].

Starting from the well-known results in [10], the generalized principle of the superposition of displacements we shall have

$$u_z(x,y)=-A\left\{\iint\limits_{\omega}\frac{p(\xi,\eta)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{1-h/2}}\right\}^{\gamma} \tag{1.1}$$

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$$A = c(h)K^{-\gamma}, \quad \gamma = \frac{1}{h}$$

for the vertical displacements $u_z(x, y)$ ($u_z(x, y)$ will actually be velocities rather than displacements; however, for consistency in the subsequent treatment we shall use the term "displacements") of the boundary points of the half-space due to the unknown normal contact stresses $p(x, y)$ acting in the domain Ω. In (1.1), $c(h)$ is a certain constant [10] and $c(2/3)=0$, $c(1)=1/(4\pi)$ and $c(h)>0$ when $2/3<h\leqslant 1$. We henceforth assume that the latter constraint on h is satisfied.

Now, by taking account of the known contact condition

$$-w(x,y) = \delta - f(x,y) \quad ((x,y,0)\in\omega)$$

where δ is the settlement of the punch and $f(x, y)$ is a function which characterizes its base, using (1.1) we arrive at the following integral equation in $p(x, y)$

$$\iint_\omega \frac{p(\xi,\eta)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{1-h/2}} = \left[\frac{\delta - f(x,y)}{A}\right]^h \tag{1.2}$$

$$\iint_\omega p(\xi,\eta)d\xi d\eta = P \quad (P<\infty)$$

A comparison of the asymptotic forms of the left- and right-hand sides of (1.2) when $x^2+y^2\to\infty$ yields the relation

$$f(x,y) \approx \delta - AP^\gamma (x^2+y^2)^{1/2-\gamma} \quad (x^2+y^2\to\infty)$$

from which δ is actually determined.

Next, we change to dimensionless variables in the integral equation (1.2) by replacing x, y; ξ, η by ax, ay; $a\xi$, $a\eta$, respectively. As a result, we obtain the equation

$$\iint_{\omega_0} \frac{\varphi(\xi,\eta)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\mu+1/2}} = g(x,y) \tag{1.3}$$

$$\omega_0 = \{z=0,\ -\infty<x<\infty,\ -1<y<1\}$$

$$\varphi(x,y) = A^h p(ax,ay), \quad g(x,y) = [\delta_0 - f_0(x,y)]^h$$

$$\delta_0 = \delta/a, \quad f_0(x,y) = f(ax,ay)/a, \quad \mu = (1-h)/2$$

Hence, the solution of the contact problem in question reduces to solving the integral equation (1.3). The generalized displacements $w(x, y)$ of the boundary points of the half-space outside the strip punch are, according to (1.1), given by the formula

$$w(x,y) = \iint_{\omega_0} \frac{\varphi(\xi,\eta)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\mu+1/2}}, \quad (x,y)\in\Pi\setminus\omega_0 \tag{1.4}$$

where

$$\Pi = \{z=0,\ -\infty<x,y<\infty\}, \quad w(x,y) = [-a^{-1}u_z(x,y)]^h$$

Using a Fourier integral transformation with respect to the variable x and a formula in [11, p. 443, 3.773.6], Eq. (1.3) can be reduced to the following equivalent one-dimensional integral equation

$$\int_{-1}^{1} \frac{K_\mu(|s||y-\eta|)}{|y-\eta|^\mu} \varphi_s(\eta) d\eta = f_s(y) \tag{1.5}$$

$$\varphi_s(y) = \int_{-\infty}^{\infty} \varphi(x,y) e^{isx} dx, \quad g_s(y) = \int_{-\infty}^{\infty} g(x,y) e^{isx} dx$$

$$f_s(y) = F_s(\mu) g_s(y), \quad F_s(\mu) = \pi^{-1/2} 2^{\mu-1} \Gamma(\mu + 1/2) |s|^{-\mu}$$

where $K_\mu(y)$ is a MacDonald function. Relationship (1.4) now takes the form

$$w_s(y) = \int_{-1}^{1} \frac{K_\mu(|s||y-\eta|)}{|y-\eta|^\mu} \psi_s(\eta) d\eta, \quad |y| > 1 \tag{1.6}$$

$$\psi_s(y) = \frac{\varphi_s(y)}{F_s(\mu)}, \quad w_s(y) = \int_{-\infty}^{\infty} w(x,y) e^{isx} dx$$

2. Let us construct the solution of the integral equation (1.3), or of Eq. (1.5) which is equivalent to it, using generalized potential theory methods. For this purpose, we will introduce the generalized potential

$$U(x,y,z) = \iint_{\omega_0} \frac{\varphi(\xi,\eta) d\xi d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{\mu+1/2}} \quad (|\mu| < 1/2) \tag{2.1}$$

into the treatment.

Then, using the well-known results in [3, 6, 7], it can be shown that the integral equation (1.3) is equivalent to the following boundary-value problem for the domain which is external to ω_0

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \frac{2\mu}{z}\frac{\partial U}{\partial z} = 0 \quad ((x,y,z) \overline{\in} \omega_0) \tag{2.2}$$

$$U(x,y,z)|_{z=0} = g(x,y) \quad ((x,y,0) \in \omega_0)$$

$$U(x,y,z) \approx Q r^{-1-2\mu} \quad (r \to \infty, \quad r = \sqrt{x^2+y^2+z^2}, \quad Q = PA^h / a^2)$$

The differential equation in (2.2) is called the Euler–Poisson–Darboux differential equation and some properties of this equation have been indicated in [12].

After the solution of the boundary-value problem (2.2) has been constructed, the density of the sources, that is, the solution of Eq. (3.3) is determined using the formula [6]

$$-2\pi\varphi(x,y) = \operatorname{sgn} z \lim_{z \to 0} |z|^{2\mu} \frac{\partial U}{\partial z} ((x,y,0) \in \omega_0) \tag{2.3}$$

Using a Fourier transformation with respect to the variable x, the boundary-value problem (2.2) can be reduced to the following equivalent boundary-value problem

$$\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{2\mu}{z}\frac{\partial V}{\partial z} - s^2 V = 0 \quad ((y,z) \overline{\in} L) \tag{2.4}$$

$$V(y,z,s)|_{z=0} = g_s(y) \quad ((y,0) \in L)$$

$$V(y,z,s) \to 0 \quad (y^2 + z^2 \to \infty)$$

for the whole yz plane with a cut along the length $L=\{z=0,\ -1\leqslant y\leqslant 1\}$ and, moreover, by (2.1)

$$V(y,z,s)=U_s(y,z)=\int_{-\infty}^{\infty}U(x,y,z)e^{isx}dx= \tag{2.5}$$

$$=\frac{\sqrt{\pi}\,|s|^{\mu}}{2^{\mu-1}\Gamma(\mu+\frac{1}{2})}\int_{-1}^{1}\frac{K_\mu(|s|\sqrt{(y-\eta)^2+z^2})}{[(y-\eta)^2+z^2]^{\mu/2}}\varphi_s(\eta)d\eta$$

At the same time, formula (2.3) acquires the form

$$-2\pi\varphi_s(y)=\operatorname{sgn} z\lim_{z\to 0}|z|^{2\mu}\frac{\partial V}{\partial z}((y,0)\in L) \tag{2.6}$$

In the final analysis we obtain that the integral equation (1.5), in turn, is equivalent to the boundary-value problem (2.4) and that their solutions are related to one another by relationship (2.6).

We will construct the solution of boundary-value problem (2.4) by the method of separation of variables. Let us put

$$V(y,z,s)=|z|^{-\mu}\,W(y,z,s) \tag{2.7}$$

and introduce the elliptic coordinates [13]

$$w=y+iz=\operatorname{ch}\zeta,\quad \zeta=u+i\upsilon,\quad u\geqslant 0,\ -\pi<\upsilon\leqslant\pi \tag{2.8}$$

$$y=\operatorname{ch}u\cos\upsilon,\quad z=\operatorname{sh}u\sin\upsilon \tag{2.9}$$

Using the conformal mapping (2.8), the complex plane w with the cut L is mapped onto the half-strip $\Pi_+=\{u\geqslant 0,\ -\pi<\upsilon\leqslant\pi\}$ and the line $u=0$ corresponds to twice the enveloped piece L of the w plane.

Taking (2.9) into account, we put

$$W(y,z,s)=W(\operatorname{ch}u\cos\upsilon,\operatorname{sh}u\sin\upsilon,s)=W_0(u,\upsilon)=F(u)G(\upsilon) \tag{2.10}$$

Using well-known results [13] and after some elementary algebra, we reduce the partial differential equation in (2.4) to the following two ordinary differential equations

$$F''-[\alpha+2q\operatorname{ch}2u-\mu(1-\mu)\operatorname{sh}^{-2}u]F=0,\quad 0\leqslant u<\infty \tag{2.11}$$

$$G''+[\alpha+2q\cos 2\upsilon+\mu(1-\mu)\sin^{-2}\upsilon]G=0,\quad -\pi<\upsilon\leqslant\pi\ (q=s^2/4) \tag{2.12}$$

where α is a separation parameter. It can be seen that, if one formally puts $\upsilon=iu$, Eq. (2.12) reduces to Eq. (2.11). We can therefore confine ourselves to one of them, Eq. (2.12), for example. This equation must be considered subject to the periodicity condition

$$G(\upsilon)=G(\upsilon+2\pi) \tag{2.13}$$

Equation (2.12) is transformed by the substitution $G(\upsilon)=\sqrt{|\sin\upsilon|}H(\upsilon)$ into the differential equation for spheroidal wave functions [13, p. 170]

$$\frac{d^2H}{d\upsilon^2}+\operatorname{ctg}\upsilon\frac{dH}{d\upsilon}+[\lambda+4\theta\sin^2\upsilon-\kappa^2\sin^{-2}\upsilon]H=0 \tag{2.14}$$

$$\lambda=\alpha-\tfrac{1}{4}+2q,\quad \theta=-q=-s^2/4,\quad \kappa=\tfrac{1}{2}-\mu$$

Equation (2.14) and its solution have been treated in detail in [14] as well as in [15].

The spheroidal wave functions

$$P^\kappa s_\nu(\cos\upsilon,\theta) = \sum_{r=-\infty}^{\infty}(-1)^r a^\kappa_{\nu,r}(\theta) P^\kappa_{\nu+2r}(\cos\upsilon) \tag{2.15}$$

$$Q^\kappa s_\nu(\cos\upsilon,\theta) = \sum_{r=-\infty}^{\infty}(-1)^r a^\kappa_{\nu,r}(\theta) Q^\kappa_{\nu+2r}(\cos\upsilon) \quad (0<\upsilon<\pi)$$

are the two linearly independent solutions of (2.14). In (2.15), ν is the characteristic index of Eq. (2.14), where $\lambda=\lambda^\kappa_\nu(\theta)$ and $\lambda^\kappa_\nu(0)=\nu(\nu+1)$, $P^\kappa_\nu(x)$ and $Q^\kappa_\nu(x)$ are Legendre functions of the first and second kind, respectively, and the coefficients $a^\kappa_{\nu,r}(\theta)$ are determined from three-term recurrence relationships [13, p. 171, (9)]. These coefficients are chosen such that

$$a^\kappa_{\nu,0}(\theta) = a^\kappa_{-\nu-1,0}(\theta) = a^{-\kappa}_{\nu,0}(\theta), \quad a^\kappa_{\nu,0}(0)=1$$

Then [13]

$$a^\kappa_{\nu,r}(\theta) = a^\kappa_{-\nu-1,-r}(\theta) = \frac{\Gamma(\nu-\kappa+2r+1)}{\Gamma(\nu-\kappa+1)}\cdot\frac{\Gamma(\nu+\kappa+1)}{\Gamma(\nu+\kappa+2r+1)} a^{-\kappa}_{\nu,r}(\theta) \tag{2.16}$$

In order to determine ν and to separate one of the solutions (2.15), we note that, according to the second relationship of boundary-value problem (2.4) where, as a consequence of (2.9), it is necessary to put $y=\cos\upsilon$ in the cut along L, the required potential must be an even function of υ. This condition, together with the periodicity condition (2.13) and Eq. (2.14) constitute a Sturm–Liouville boundary-value problem. Using the latter and the well-known trigonometric expansions of the functions $P^\kappa_\nu(\cos\upsilon)$ and $Q^\kappa_\nu(\cos\upsilon)$ [16, p. 147], we immediately obtain

$$\nu = n-\kappa \quad (n=0,1,2,\ldots) \tag{2.17}$$

and the first solution of (2.15) has to be taken. However, as follows from (2.16) and (2.17), in the case under discussion

$$a^\kappa_{\nu,r}(\theta)=0 \quad \text{when} \quad r \leqslant -[n/2]-1$$

Consequently, the first series in (2.15) is terminated to the left at a certain term. allowing for this and the well-known relationship between Legendre functions of the first kind and Gegenbauer polynomials $C^\mu_n(x)$ [16, p. 177, (4)], we finally find

$$G^\kappa_n(\upsilon) = \sqrt{\sin\upsilon}\, Ps^\kappa_{n-\kappa}(\cos\upsilon,\theta) = \frac{2^\kappa(\sin\upsilon)^{1/2-\kappa}\Gamma(1-2\kappa)}{\Gamma(1-\kappa)}\sum_{r=0}^{\infty}\frac{(-1)^{r-m}(2r)!}{\Gamma(2r+1-2\kappa)}\times \tag{2.18}$$

$$\times a^\kappa_{2m-\kappa,r-m}(\theta) C^{1/2-\kappa}_{2r}(\cos\upsilon) \quad (n=2m)$$

$$G^\kappa_n(\upsilon) = \sqrt{\sin\upsilon}\, Ps^\kappa_{n-\kappa}(\cos\upsilon,\theta) = \frac{2^\kappa(\sin\upsilon)^{1/2-\kappa}\Gamma(1-2\kappa)}{\Gamma(1-\kappa)}\sum_{r=0}^{\infty}\frac{(-1)^{r-m}(2r+1)!}{\Gamma(2r+2-2\kappa)}\times$$

$$\times a^\kappa_{2m+1-\kappa,r-m}(\theta) C^{1/2-\kappa}_{2r+1}(\cos\upsilon) \quad (n=2m+1)$$

$$(\kappa = 1/2-\mu, \quad m=0,1,2,\ldots, 0<\upsilon<\pi)$$

The functions $G^\mu_n(\upsilon)$ are evenly continued in the interval $-\pi<\upsilon<0$.

Since the coefficients $a^\kappa_{\nu,r}(\theta)$ satisfy three-term homogeneous recurrence relationships, they can be found from these, apart from a constant factor. In order to determine this factor, we normalize the functions $G_n^\mu(\upsilon)$ in the following manner [14] (this normalization differs from that used in [14] when $n=0$)

$$\int_0^\pi [G_n^\kappa(\upsilon)]^2 d\upsilon = h_n, \quad h_n = \begin{cases} n!/[\Gamma(n+1-2\kappa)(n+1/2-\kappa)] \\ n=1,2,\dots \\ 1/\Gamma(2-2\kappa), \quad n=0 \end{cases}$$

Making use of the orthogonality of Gegenbauer polynomials, this relationship can be written in the form

$$\sum_{r=0}^{\infty} \gamma_{nr}^{\kappa}[a_{n-\kappa,r-[n/2]}^{\kappa}(\theta)]^2 = h_n \tag{2.19}$$

$$\gamma^{\kappa}{}_{nr} = \begin{cases} f(2r), \quad n=2m \\ \qquad\qquad\qquad (m=0,1,2,\dots) \\ f(2r+1), \quad n=2m+1 \end{cases}$$

$$f(x) = (x)![\Gamma(x+1-2\kappa)\Gamma(x+1/2-\kappa)]^{-1}$$

Only the value of the absolute magnitude of the above-mentioned factor is determined using (2.19). In order to determine its sign, we put $a_{\nu,0}^{\kappa}(\theta)>0$. We can also put

$$(-1)^{[n/2]}[G_n^\kappa(\upsilon)(\sin\upsilon)^{\kappa-1/2}]|_{\upsilon=0} > 0$$

which yields

$$\sum_{r=0}^{\infty}(-1)^r a_{\nu,r-[n/2]}^{\kappa}(\theta) > 0, \quad \nu = n-\kappa \tag{2.20}$$

In the limiting case when $\mu \to 0$, it can be shown that the sequence $a_{\nu,r}^{\kappa}(\theta)>0$ is not terminated and the three-term recurrence relationships for determining them reduce to the corresponding relationships for the coefficients of the periodic Mathieu functions $\mathrm{ce}_n(\upsilon,\ \theta)$ [13, p. 155]. Here

$$G_n^{1/2}(\upsilon) = (-1)^{[n/2]}\sqrt{2/\pi}\,\mathrm{ce}_n(\upsilon,\theta) \tag{2.21}$$

and the normalization conditions (2.19) and (2.20) reduce to the well-known conditions for the normalization of the functions $\mathrm{ce}_n(\upsilon,\ \theta)$ [13, p. 156].

We now consider Eq. (2.11). The unique solution of this equation, which is bounded on the semi-axis $0 \leqslant u < \infty$ and vanishes at infinity has the form [13–15]

$$F_n^\kappa(u) = \sqrt{\mathrm{sh}\,u}\, S_{n-\kappa}^{\kappa(3)}(\mathrm{ch}\,u,\theta) \quad (0 \leqslant u < \infty) \tag{2.22}$$

where $S_\nu^{\kappa(3)}(z,\ \theta)$ is a spheroidal wave function of the third kind. The following representation can be obtained for this function

$$S_{2m-\kappa}^{\kappa(3)}(\mathrm{ch}\,u,\theta) = l_{2m}^{\kappa}(\theta)\frac{\mathrm{th}^{-\kappa}(u)}{\sqrt{\mathrm{ch}\,u}} \times$$

$$\times \sum_{r=0}^{\infty}(-1)^r a_{2m-\kappa,r-m}^{\kappa}(\theta) K_{\mu+2r}(2\sqrt{q}\,\mathrm{ch}\,u) \quad (0 \leqslant u < \infty) \tag{2.23}$$

$$S_{2m+1-\kappa}^{\kappa(3)}(\mathrm{ch}\,u,\theta) = l_{2m+1}^{\kappa}(\theta)\frac{\mathrm{th}^{-\kappa}(u)}{\sqrt{\mathrm{ch}\,u}} \sum_{r=0}^{\infty}(-1)^r a_{2m+1-\kappa,r-m}^{\kappa}(\theta) K_{\mu+2r+1}(2\sqrt{q}\,\mathrm{ch}\,u)$$

$$l_{2m}^{\kappa}(\theta)=\frac{e^{-i\pi(1-\kappa/2)}}{(\pi^2 q)^{1/4}}s_{2m-\kappa}^{\kappa}(\theta)$$

$$l_{2m+1}^{\kappa}(\theta)=\frac{e^{-i\pi(3-\kappa)/2}}{(\pi^2 q)^{1/4}}s_{2m+1-\kappa}^{\kappa}(\theta)$$

$$s_{n-\kappa}^{\kappa}(\theta)=\left[\sum_{r=0}^{\infty}(-1)^{r-[n/2]}a_{n-\kappa,r-[n/2]}^{\kappa}(\theta)\right]^{-1}$$

Hence, by (2.7) and (2.10), the boundary-value problem (2.4) possesses a normal solution of the form

$$V(y,z,s)=(\operatorname{sh}u\sin\upsilon)^{-\mu}F_n^{\kappa}(u)G_n^{\kappa}(\upsilon)\quad(0\leqslant u<\infty,\ \ 0<\upsilon<\pi) \tag{2.24}$$

The functions $F_n^{\kappa}(u)$ are expressed by formulae (2.22) and (2.23), and the functions $G_n^{\kappa}(\upsilon)$ are expressed by formulae (2.18) and the variables y, z and u, υ are related by (2.9).

In order to calculate the density of the sources which corresponds to the potential (2.24) in explicit form, we make use of well-known formulae in [13 p. 173, (20) and p. 174, (28)] and using these, we obtain the representation

$$\begin{aligned}&S_\nu^{\kappa(3)}(\operatorname{ch}u,\theta)=ie^{i\pi\mu}(\pi\sin\pi\mu)^{-1}[e^{i\pi\mu}\tilde{K}_{-\nu-1}^{\kappa}(\theta)Qs_\nu^{\kappa}(\operatorname{ch}u,\theta)+\\&+\sin(2\pi\mu)e^{-2i\pi\mu}K_\nu^{\kappa}(\theta)Qs_{-\nu-1}^{\kappa}(\operatorname{ch}u,\theta)]\quad 0\leqslant u<\infty\\&\nu=n-\kappa,\quad \kappa=1/2-\mu,\quad \theta=-q=-s^2/4\end{aligned} \tag{2.25}$$

where $K_\nu^{\kappa}(\theta)$ is the known coefficient of the relation between the functions $S_\nu^{\kappa(1)}(z,\,0)$ and $Q_{s-\nu-1}^{\kappa}(z,\,\theta)$ [13, p. 175, (29) when $k=0$], $\tilde{K}_{-\nu-1}^{\kappa}(\theta)=\lim_{\nu+\kappa\to n}K_{-\nu-1}^{\kappa}(\theta)$, and the functions $Qs_\nu^{\kappa}(\operatorname{ch}u,\ \theta)$ and $Qs_{-\nu-1}^{\kappa}(\operatorname{ch}u,\ \theta)$ are expressed by the second formula in (2.15), if one formally puts $\upsilon=iu$.

We now note that, in obtaining the representation of $Qs_\nu^{\kappa}(\operatorname{ch}u,\ \theta)$ by this means, the functions $Qs_{\nu+2r}^{\kappa}(\operatorname{ch}2u)$ become infinite when $r\leqslant-[n/2]-1$ while the corresponding coefficients $a_{\nu,r}^{\kappa}(\theta)=0$, and the series being considered is therefore not terminated. The latter leads to the need to pass to the limit. Making use of the formula in [16, p. 140, (2)]

$$Q_\nu^{\kappa}(z)=e^{2i\pi\kappa}\Gamma(\nu+\kappa+1)[\Gamma(\nu-\kappa+1)]^{-1}Q_\nu^{-\kappa}(z)$$

and using (2.16) in the expression $a_{\nu,r}^{\kappa}(\theta)Q_{\nu+2r}^{\kappa}(\operatorname{ch}u)$, we pass to the limit as $\nu+\kappa\to n$. As a result, we arrive at the representation

$$\begin{aligned}&Qs_\nu^{\kappa}(\operatorname{ch}u,\theta)=\frac{n!e^{-2i\pi\mu}}{\Gamma(n+2\mu)}\sum_{r=-\infty}^{-[n/2]-1}(-1)^{r+1}a_{\nu,r}^{\kappa}(\theta)Q_{\nu+2r}^{-\kappa}(\operatorname{ch}u)+\\&+\sum_{r=-[n/2]}^{\infty}(-1)^{r}a_{\nu,r}^{\kappa}(\theta)Q_{\nu+2r}^{\kappa}(chu),\quad 0\leqslant u<\infty\\&\nu=n-\kappa,\quad \kappa=1/2-\mu,\quad n=0,1,2,\ldots\end{aligned} \tag{2.26}$$

For the functions $Qs_{-\nu-1}^{\kappa}(\operatorname{ch}u,\ \theta)$, we shall have a representation in the form of the following series which is terminated from the left

$$Qs_{-\nu-1}^{\kappa}(\operatorname{ch}u,\theta)=\sum_{r=-[n/2]}^{\infty}(-1)^{r}a_{\nu,r}^{\kappa}(\theta)Q_{-\nu-1-2r}^{\kappa}(\operatorname{ch}u) \tag{2.27}$$

For subsequent purposes, it is convenient to express the functions $Q_\nu^\kappa(z)$ in (2.26) and (2.27) by the formula [16, p. 134, (34)]

$$Q_\nu^\kappa(z) = 2^{-\nu-1}(z^2-1)^{-\kappa/2} e^{i\pi\kappa}\left[\Gamma(\kappa)(z+1)^{\nu+\kappa} F\left(-\nu, -\nu-\kappa; 1-\kappa; \frac{z-1}{z+1}\right)+\right. \tag{2.28}$$

$$\left. +(z+1)^\nu (z-1)^\kappa \frac{\Gamma(\nu+\kappa+1)\Gamma(-\kappa)}{\Gamma(1+\nu-\kappa)} F\left(-\nu, -\nu+\kappa; 1+\kappa; \frac{z-1}{z+1}\right)\right]$$

where $F(a, b; c; z)$ is a Gaussian hypergeometric function.

Starting from (2.6), (2.9) and (2.24), we now obtain

$$-2\pi\varphi_s(y) = (\sin\upsilon)^{\mu-1} G_n^\kappa(\upsilon) \lim_{u\to 0}\left\{(\operatorname{sh} u)^{2\mu} \frac{d}{du}\left[(\operatorname{sh} u)^{-\mu} F_n^\kappa(u)\right]\right\} \quad (0<\upsilon<\pi) \tag{2.29}$$

Next, by using formulae (2.5), (2.23) and (2.25)–(2.29), after certain transformations we arrive at the spectral relationship

$$\int_{-1}^{1} \frac{K_\mu(|s||y-u|)}{|y-u|^\mu}(1-u^2)^{-\kappa/2} Ps_{n-\kappa}^\kappa(u,\theta)du = \lambda_n (1-y^2)^{\kappa/2} Ps_{n-\kappa}^\kappa(y,\theta) \tag{2.30}$$

$$(|\mu|<1/2,\ |y|<1,\ n=0,1,2,\ldots)$$

$$\lambda_n = (-1)^{[n/2]}\alpha_n \sin(\pi\mu) q^{(n+\mu)/2} |2s|^{-\mu} 2^{1-n} A_n^\kappa(\theta) B_n^\kappa(\theta)[E_n^\kappa(\theta)]^{-1}$$

$$\alpha_n = \begin{cases} 1, & n=2m \\ -1, & n=2m+1 \end{cases}$$

$$A_n^\kappa(\theta) = \sum_{r=0}^{[n/2]} \frac{(-1)^r a_{n-\kappa,-r}^\kappa(\theta)}{r!\Gamma(n+\mu+1-r)}\left[\sum_{r=0}^{\infty} \frac{(-1)^r a_{n-\kappa,r}^\kappa(\theta)}{r!\Gamma(1-n-\mu-r)}\right]^{-1}$$

$$B_n^\kappa(\theta) = (-1)^{[n/2]} \sum_{r=-[n/2]}^{\infty} (-1)^r a_{n-\kappa,r}^\kappa(\theta) K_{n+\mu+2r}(2\sqrt{q})$$

$$E_n^\kappa(\theta) = \frac{1}{n!}\sum_{r=-[n/2]}^{\infty} (-1)^r \frac{(n+2r)!}{\Gamma(n+2\mu+2r)} a_{n-\kappa,r}^\kappa(\theta) + \frac{1}{\Gamma(n+2\mu)} \sum_{r=-\infty}^{-[n/2]-1} (-1)^r a_{n-\kappa,r}^{-\kappa}(\theta)$$

In order to obtain a relationship related to (2.30) which is valid in rays $|y|>1$, we note that, according to (2.9), the line $\upsilon=0$ corresponds to a ray $y>1$ and the line $\upsilon=\pi$ corresponds to a ray $y<-1$. On taking account of the latter and again using (2.5), (2.23) and (2.25)–(2.29), we arrive at the relationship

$$\int_{-1}^{1} \frac{K_\mu(|s||y-u|)}{|y-u|^\mu}(1-u^2)^{-\kappa/2} Ps_{n-\kappa}^\kappa(u,\theta)du = \nu_n \operatorname{sgn} y (y^2-1)^{\kappa/2} S_{n-\kappa}^{\kappa(3)}(|y|,\theta) \tag{2.31}$$

$$(|\mu|<1/2,\ |y|>1,\ n=0,1,2,\ldots)$$

$$\nu_n = (-1)^{1+[n/2]}\beta_n \sin(\pi\mu) q^{(n+\mu)/2} |2s|^{-\mu} 2^{1-n} A_n^\kappa(\theta)[E_n^\kappa(\theta)]^{-1}$$

$$\beta_n = \begin{cases} (\pi^2 q)^{1/4} e^{-i\pi\kappa/2}, & n=2m \\ -(\pi^2 q)^{1/4} e^{i\pi(1-\kappa)/2}, & n=2m+1 \end{cases}$$

According to the Schwartz symmetry principle for analytic functions, relationships (2.30) and (2.31) can be analytically continued in the band $|\mathrm{Re}\,\mu|<1/2$.

In the limiting case when $\mu\to 0$ according to (2.21) the spectral relationship (2.30) becomes the well-known relationship in [17] and (2.31) takes the form

$$\int_{-1}^{1} K_0(|s||y-u|)\,\mathrm{ce}_n(\arccos u,-q)(1-u^2)^{-1/2}du=\mu_n c_n(y)\,\mathrm{Fek}_n(\ln|y|+\sqrt{1-y^2},-q)$$

$$|y|>1,\quad n=0,1,2,\ldots$$

$$\mu_n=-\pi\,\mathrm{ce}_n(0,-q)/\mathrm{Fe'k}_n(0,-q)$$

$$c_n(y)=\begin{cases}1, & n=2m\\ \mathrm{sgn}\,y, & n=2m+1\end{cases}$$

where $\mathrm{Fek}_n(z,\,\theta)$ is a modified Mathieu function of the third kind [13].

3. Considering now the solution of the integral equation (1.5) of the contact problem being considered, we put

$$\varphi_s(y)=(1-y^2)^{-\kappa/2}\sum_{n=0}^{\infty}X_n Ps_{n-\kappa}^{\kappa}(y,\theta)\qquad(|y|<1)\tag{3.1}$$

where the coefficients $\{X_n\}_{n=0}^{\infty}$ are unknown. We then substitute (3.1) into (1.5), interchange the order of summation and integration and make use of the spectral relationship (2.30). As a result, we shall have

$$(1-y^2)^{\kappa/2}\sum_{n=0}^{\infty}X_n\lambda_n Ps_{n-\kappa}^{\kappa}(y,\theta)=f_s(y)\quad(|y|<1)$$

Hence, on taking account of the condition for spheroidal wave functions to be orthogonal

$$\int_{-1}^{1}Ps_{n-\kappa}^{\kappa}(y,\theta)Ps_{m-\kappa}^{\kappa}(y,\theta)dy=\begin{cases}h_n, & m=n\\ 0, & m\neq n\end{cases}$$

we obtain

$$X_n=\frac{f_n}{\lambda_n h_n},\quad f_n=\int_{-1}^{1}f_s(y)(1-y^2)^{-\kappa/2}Ps_{n-\kappa}^{\kappa}(y,\theta)dy\tag{3.2}$$

Thus, the solution of Eq. (1.5) is expressed by formulae (3.1) and (3.2).

In order to determine the Fourier transform of the reduced displacements beyond the strip punch, we substitute the function $\varphi_s(y)$ from (3.1) into (1.6) and take account of relationship (2.31). Then

$$w_s(y)=\mathrm{sgn}\,y(y^2-1)^{\kappa/2}\sum_{n=0}^{\infty}\frac{\nu_n f_n}{\lambda_n h_n}S_{n-\kappa}^{\kappa(3)}(|y|,\theta)\quad(|y|>1)$$

For the particular configuration of the strip punch $g(x,\ y)=\cos sx$ $(s>0)$, the integral equation (1.5) takes the form

$$\int_{-1}^{1}\frac{K_\mu(s|y-\eta|)}{|y-\eta|^\mu}\varphi_s(\eta)d\eta=d_s$$

$$d_s=\pi^{-1/2}2^{\mu-1}\Gamma(\mu+1/2)s^{-\mu}$$

Since the solution of this equation is an even function, the coefficients in (3.1) $X_{2m+1} = 0$ ($m = 0, 1, 2, \dots$). The remaining coefficients X_{2m} are determined using formula (3.2), where it is necessary to put $n = 2m$. Using the representation (2.18), we obtain

$$f_{2m} = (-1)^m \frac{\sqrt{\pi}\, 2^{\kappa} d_s}{\Gamma(3/2 - \kappa)} a^{\kappa}_{2m-\kappa,-m}(\theta) \quad (m = 0, 1, 2, \dots)$$

Note that, in the limiting case when $\mu \to 0$, the results described here for a strip punch reduce to the well-known results in [18].

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